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Multifractal structure of a general subordinator

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Abstract

The object of this paper is to obtain the multifractal structure for the local time of a general Lévy process which hits points. All such local time measures can be represented as the occupation measure of a subordinator. The thick points in the spectrum were investigated in a recent paper by Marsalle (1999. Ann. Probab. 27, 150–165). In this paper we obtain the spectrum for thin points, obtaining analogues of our results (Hu and Taylor, 1997, Stochastic Process. Appl. 66, 283–299) for the stable subordinator. We obtain our precise dimension results for exceptional sets in time rather than in space.

Keywords: Multifractal structure; General subordinator; Hausdorff dimension; Thin points

1. Introduction

We consider only subordinators on **R** – that is, a monotone increasing process $X(t)$ with stationary, independent increments. It is well known that the process X is determined by the exponent $g(r)$ defined by

$$E\mathrm{e}^{-r[X(s+t)-X(s)]} = \mathrm{e}^{-tg(r)}.$$

In the stable case there is an index $\alpha \in (0,1)$ such that $g(r) = r^{\alpha}$ and much is known about the sample trajectory $X(t) = X(t,\omega)$, including the multifractal structure of the occupation measure defined by

$$\mu(A) = |\{t \in [0,1]\colon X(t) \in A\}|,$$

which was clarified in Hu and Taylor (1997).

The object of the present paper is to establish analogues of the results in Hu and Taylor (1997) for a general subordinator. Some of the results will require mild regularity conditions on the growth of $g(r)$. The scaling properties of $g(r)$ as $r \to \infty$ influence the corresponding scaling of $\mu(x-r,x+r)$ as $r \downarrow 0$. In general $\log g(r)/\log r$ will vary in

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a range $\subset [0,1]$ as $r \to \infty$, and this causes the expected value of $\log \mu(x-r,x+r)/\log r$ to oscillate as $r \downarrow 0$. If we define indices σ, β by

$$\sigma = \liminf_{r\to\infty} \frac{\log g(r)}{\log r}, \qquad \beta = \limsup_{r\to\infty} \frac{\log g(r)}{\log r},$$

we always have $0 \leqslant \sigma \leqslant \beta \leqslant 1$ (Pruitt and Taylor, 1996). Fristedt and Pruitt (1971) gave a construction for a gauge function $\phi(s)$ such that the Hausdorff measure $\phi - mX([0,1]) = c$ a.s. (c is a constant). Fristedt and Taylor (1992) showed that there is a special class of subordinators with an exact packing gauge function and found integral tests to determine whether $\psi - pX([0,1]) = 0$ or ∞ a.s. for subordinators not in this class. From these precise results we can deduce that

$$\dim S = \sigma, \qquad \operatorname{Dim} S = \beta \text{ a.s.},$$

where $S = \overline{X[0,1]}$ is the support of the random measure μ and dim, Dim denote the Hausdorff and packing dimensions, respectively. It is easy to deduce that for μ a.e. $x \in S$

$$\underline{d}(\mu,x) := \liminf_{r\to 0} \frac{\log \mu(x-r,x+r)}{\log r} = \sigma,$$

$$\bar{d}(\mu,x) := \limsup_{r\to 0} \frac{\log \mu(x-r,x+r)}{\log r} = \beta.$$

The main result obtained in this paper is that, with probability 1, we have

$$\underline{d}(\mu,x) = \sigma \quad \text{for all } x \in S, \tag{1.1}$$

$$\beta \leqslant \bar{d}(\mu,x) \leqslant 2\beta \quad \text{for all } x \in S. \tag{1.2}$$

In order to get a spectrum, we consider $A_\alpha = \{x \in S\colon \bar{d}(\mu,x) = \alpha\}$. It is clear that we can not use the standard multifractal formalism when $\sigma < \beta$ because the spectrum for $\underline{d}$ is different to that for $\bar{d}$. The spectrum was found in Hu and Taylor (1997) for a stable subordinator, but in that paper we could use the fact that

$$\dim X(T) = \alpha \dim T \quad \text{for every Borel T},$$

while Hawkes and Pruitt (1974) showed that, for a general subordinator, the uniform result

$$\sigma \dim T \leqslant \dim X(T) \leqslant \beta \dim T,$$

is best possible. This means that the methods of Hu and Taylor (1997) will determine $f(\alpha)$ only in the case $\sigma = \beta$. To get over this we use a function $h(r)$ which follows the oscillations of $g(r)$ such that, with probability 1,

$$\lim_{r\to 0} \frac{\log \mu(x-r,x+r)}{\log h(r)} = 1 \quad \text{for } \mu \text{ a.e. } x. \tag{1.3}$$

For this function $h(r)$ we prove that

$$\liminf_{r\to 0} \frac{\log \mu(x-r,x+r)}{\log h(r)} = 1 \quad \text{for all } x, \tag{1.4}$$

$$1 \leqslant \limsup_{r\to 0} \frac{\log \mu(x-r,x+r)}{h(r)} \leqslant 2 \quad \text{for all } x. \tag{1.5}$$

To consider the exceptional set of times where (1.3) fails for $x = X(t)$, we define the sets T_γ, U_γ by

$$T_\gamma = \left\{ t \in [0,1]\colon\ \limsup_{r\to 0} \frac{\log \mu(X(t)-r, X(t)+r)}{\log h(r)} \geqslant \gamma \right\}, \tag{1.6}$$

$$U_\gamma = \left\{ t \in [0,1]\colon\ \limsup_{r\to 0} \frac{\log \mu(X(t)-r, X(t)+r)}{\log h(r)} = \gamma \right\}. \tag{1.7}$$

We prove that with probability 1, $T_\gamma = \emptyset$ for $\gamma > 2$ and, for $1 \leqslant \gamma \leqslant 2$,

$$\dim U_\gamma = \dim T_\gamma = \frac{2}{\gamma} - 1, \tag{1.8}$$

$$\mathrm{Dim}\, U_\gamma = \mathrm{Dim} T_\gamma = 1, \tag{1.9}$$

and $U_2 = T_2$ is non-empty. We are not able to determine the spectrum for $X(U_\gamma)$. We remark that the spectrum for "thick points" of the occupation measure μ was found in Shieh and Taylor (1998) for the stable case and in Marsalle (1999) for a general subordinator. Our paper is structured as follows. In Section 2 we collect the definitions and results we need to use. Section 3 contains the proofs of (1.3) and (1.4) and we obtain (1.1). While Section 4 gives a proof for (1.2). The final section completes the determination of the spectrum (1.7).

2. Preliminaries

Let $X(t)$ be a subordinator with $X(0) = 0$. The Laplace transform of $X(t)$ is

$$E(\exp(-uX(t)) = \exp(-tg(u)), \tag{2.1}$$

where $g(u) = \int_0^\infty (1 - \mathrm{e}^{-ux})\nu(\mathrm{d}x)$, ν is the Lévy measure of $X(t)$ satisfying $\int_0^\infty [x/(1+x)]\nu(\mathrm{d}x) < \infty$; $g(u)$ is increasing. Following Fristedt and Pruitt we define

$$\eta(u) = g^{-1}(u), \quad h_\gamma(u) = \frac{\log|\log u|}{\eta(\gamma u^{-1}\log|\log u|)}, \quad \bar{h}_\gamma(u) = \frac{|\log u|}{\eta(\gamma u^{-1}|\log u|)} \quad (\gamma > 0),$$

$$\bar{f}_\gamma(u) = \bar{h}_\gamma^{-1}(u), f_\gamma(u) = h_\gamma^{-1}(u), \quad \text{especially } \bar{f} = \bar{f}_1, \quad f = f_1. \tag{2.2}$$

There are two important indices of a subordinator

$$\sigma = \sup\{\alpha : u^{-\alpha} g(u) \to \infty \text{ as } u \to \infty\},$$

$$\beta = \inf\{\alpha : u^{-\alpha} g(u) \to 0 \text{ as } u \to \infty\}. \tag{2.3}$$

According to the following inequalities from Horowitz (1968):

$$\mathrm{e}^{-1}\lambda^{1-\gamma} \int_0^{1/\lambda} \nu(x,\infty)\,\mathrm{d}x \leqslant \lambda^{-\gamma} g(\lambda)$$

$$= \lambda^{1-\gamma} \int_0^\infty \mathrm{e}^{-\lambda x}\nu(x,\infty)\,\mathrm{d}x \leqslant (1+\mathrm{e}^{-1})\lambda^{1-\gamma} \int_0^{1/\lambda} \nu(x,\infty)\,\mathrm{d}x, \quad 0 < \gamma < 1, \tag{2.4}$$

we can deduce the equivalent definitions for σ and β:

$$\sigma = \sup\{\gamma : x^{\gamma} v(x,\infty) \to \infty \text{ as } x \to 0\}, \tag{2.5}$$

$$\beta = \inf\{\gamma : x^{\gamma} v(x,\infty) \to 0 \text{ as } x \to 0\}. \tag{2.6}$$

We only show (2.5). Let $\sigma' = \sup\{\gamma : x^{\gamma} v(x,\infty) \to \infty \text{ as } x \to 0\}$. If $\gamma \leqslant \sigma'$, then $x^{\gamma} v(x,\infty) \to \infty$ as $x \to 0$. Note that

$$\lambda^{1-\gamma} \int_0^{1/\lambda} v(x,\infty)\,\mathrm{d}x \geqslant \lambda^{1-\gamma} v\left(\frac{1}{\lambda},\infty\right) \cdot \frac{1}{\lambda} = \left(\frac{1}{\lambda}\right)^{\gamma} v\left(\frac{1}{\lambda},\infty\right),$$

so by (2.4) we have $\lambda^{-\gamma} g(\lambda) \to \infty$ as $\lambda \to \infty$. The definition of σ gives that $\gamma \leqslant \sigma$. Thus $\sigma' \leqslant \sigma$.

On the other hand, for $\gamma' < \sigma$, we have $\lambda^{-\gamma'} g(\lambda) \to \infty$ as $\lambda \to \infty$. Since

$$\int_0^{1/\lambda} v(x,\infty)\,\mathrm{d}x = \sum_{n\geqslant 1} \int_{1/(2^n\lambda)}^{1/(2^{n-1}\lambda)} v(x,\infty)\,\mathrm{d}x \leqslant \sum_{n\geqslant 1} \frac{1}{2^n\lambda} v\left(\frac{1}{2^n\lambda},\infty\right),$$

so using (2.4) again, one has $\lambda^{-\gamma'} \sum_{n\geqslant 1} 1/2^n v(1/2^n\lambda,\infty) \to \infty$ as $\lambda \to \infty$. Thus there exists a n_0 such that

$$\left(\frac{1}{\lambda}\right)^{\gamma'} v\left(\frac{1}{2^{n_0}\lambda},\infty\right) \to \infty \quad \text{as } \lambda \to \infty.$$

This implies $x^{\gamma'} v(x,\infty) \to \infty$ as $x \to 0$. Therefore $\gamma' \leqslant \sigma'$, which completes the proof of (2.5). In a similar way one can prove (2.6).

Lemma 2.1. *For any* $u > 0$,

$$\frac{\mathrm{e}^{-tg(u)} - \mathrm{e}^{-ua}}{1 - \mathrm{e}^{-ua}} \leqslant P(X(t) < a) \leqslant \mathrm{e}^{-tg(u)+ua}$$

and

$$P(X(t) \geqslant a) \leqslant \frac{1 - \mathrm{e}^{-tg(u)}}{1 - \mathrm{e}^{-ua}} \leqslant \frac{tg(u)}{1 - \mathrm{e}^{-ua}}.$$

See Fristedt and Pruitt (1971).

Lemma 6.1 in Pruitt and Taylor (1969) and Lemma 6 in Fristedt and Pruitt (1971) give

Lemma 2.2. *Let* N_k *denote the number of intervals of the form* $[j2^{-n},(j+1)2^{-n}]$ *hit by* $X(t)$, *then given* $\varepsilon > 0$,

$$EN_k \leqslant c2^{k(\beta+\varepsilon)}, \quad c \text{ is a constant.} \tag{2.7}$$

In order to prove (1.1) we need the result equivalent to that in Fristedt (1979) for the stable case. We cannot obtain this without imposing the following regularity condition on the growth of $g(r)$. We say that g is λ-smooth if

$$1 < \liminf_{x\to\infty} \frac{g(\lambda x)}{g(x)} \leqslant \limsup_{x\to\infty} \frac{g(\lambda x)}{g(x)} < \lambda. \tag{2.8}$$

We remark that condition (2.8) for a fixed $\lambda > 1$ does not imply that $\sigma = \beta$; it does require that the power law for $g(r)$ as $r \to \infty$ changes slowly.

Lemma 2.3. *If g is λ-smooth for some $\lambda > 1$, then with probability one*

$$\forall t \geqslant 0, \quad \limsup_{r \to 0} \frac{\mu(X(t)-r, X(t)+r)}{1/g(r)} > 0.$$

See Theorem B in Marsalle (1999).

3. Typical behaviour

Let $g, h_\gamma, \eta, f_\gamma$ be defined as in Section 2. If $\sigma > 0$, there exists a $\delta > 0$ such that $g(u) \geqslant u^\delta$ for u sufficiently large, then by Theorem 1 in Fristedt and Pruitt (1971), we have

$$\liminf_{t \to 0} \frac{X(t)}{h_\gamma(t)} = c \text{ a.s.,} \quad c \text{ is a constant,} \quad \gamma > 1, \tag{3.1}$$

from which we can deduce that there exists a constant c_1 such that for each $s \in [0,1]$

$$\liminf_{t \downarrow 0} \frac{|X(s+t) - X(s)| \wedge |X(s) - X(s-t)|}{h_\gamma(t)} = c_1 \text{ a.s.} \tag{3.2}$$

Therefore for each $x = X(t)$,

$$\limsup_{r \downarrow 0} \frac{\mu B(x,r)}{f_\gamma(r)} = c_2 \text{ a.s.,} \quad c_2 \text{ a constant.} \tag{3.3}$$

Since $f_\gamma(r) \leqslant r^{\sigma - \varepsilon}$($\varepsilon$ is small) for r sufficiently small and $f_\gamma(r) \geqslant r^{\sigma+\varepsilon}$ for infinitely many small r's, using (3.3) we have

$$\sigma - \varepsilon \leqslant \underline{d}(\mu, x) \leqslant \sigma + \varepsilon \text{ a.s.}$$

Letting ε tend to zero through a countable sequence, one has that $\forall x \in X([0,1])$

$$\underline{d}(\mu, x) = \sigma \text{ a.s.}$$

When $\sigma = 0$, by the definition of σ, we know that

$$\text{for any } \varepsilon > 0, \quad g(u) \leqslant u^\varepsilon, \quad \text{when} \quad u \text{ is large.} \tag{3.4}$$

Since for a fixed s

$$\bigcap_{n=m}^{2m} \left\{ X\left(s + \frac{1}{2^n}\right) - X(s) > \left(\frac{1}{2^n}\right)^{1/\varepsilon} \right\}$$

$$\subset \left\{ \bigcap_{n=m}^{2m} \left\{ X\left(s + \frac{1}{2^n}\right) - X\left(s + \frac{1}{2^{n+1}}\right) > \left(\frac{1}{2^{n+1}}\right)^{1/\varepsilon} \right\} \right.$$

$$\left. \cup \bigcup_{n=m}^{2m} \left\{ X\left(s + \frac{1}{2^{n+1}}\right) - X(s) > \left(\frac{1}{2^n}\right)^{1/\varepsilon} - \left(\frac{1}{2^{n+1}}\right)^{1/\varepsilon} \right\} \right\},$$

so using Lemma 2.1 we obtain, for any $u_n > 0$ and $t_n > 0$,

$$P\left(\bigcap_{n=m}^{2m}\left\{X\left(s+\frac{1}{2^n}\right)-X(s)>\left(\frac{1}{2^n}\right)^{1/\varepsilon}\right\}\right)$$

$$\leqslant \exp\left(-\sum_{n=m}^{2m}\frac{\mathrm{e}^{-1/(2^{n+1})g(u_n)}-\mathrm{e}^{-u_n(1/(2^{n+1}))^{1/\varepsilon}}}{1-\mathrm{e}^{-u_n(1/(2^{n+1}))^{1/\varepsilon}}}\right)+\sum_{n=m}^{2m}\frac{1/(2^{n+1})g(t_n)}{1-\mathrm{e}^{-t_n[(1/2^n)^{1/\varepsilon}-(1/(2^{n+1}))^{1/\varepsilon}]}}$$

$$:= A + B.$$

Take $u_n = (2^{n+1})^{1/\varepsilon}(\frac{1}{2}\ln(n+1))^{1/\varepsilon}$ and $t_n = (2^n)^{1/\varepsilon}$, (3.4) ensures that for large m, $g(u_n) < u_n^\varepsilon$ and $g(t_n) < t_n^\varepsilon$, when $n \geqslant m$, m is large. Hence

$$A \leqslant \mathrm{e}^{-m^{1/2}} \quad \text{and} \quad B \leqslant 2\left(\frac{1}{2^m}\right)^{1-\varepsilon}.$$

Therefore there exists a $\delta > 0$, for any M large enough,

$$P\left(\bigcap_{n=M}^{2M}\left\{X\left(s+\frac{1}{2^n}\right)-X(s)>\left(\frac{1}{2^n}\right)^{1/\varepsilon}\right\}\right)\leqslant \mathrm{e}^{-M^\delta}.$$

This implies, for any $\varepsilon > 0$,

$$\liminf_{t\to 0}\frac{|X(s+t)-X(s)|\wedge|X(s)-X(s-t)|}{t^{1/\varepsilon}}=0.$$

From this result we easily deduce that

$$\liminf_{r\to 0}\frac{\log \mu B(x,r)}{\log r}=0,$$

which means $\underline{d}(\mu, X(s)) = 0$ a.s.

Using a Fubini argument we obtain the following theorem, valid for any subordinator:

Theorem 3.1. *With probability one, $\underline{d}(\mu, x) = \sigma$ for μ-a.e. x.*

According to Theorems 2 and 3 in Fristedt and Taylor (1992), in order to find out the behaviour of $\limsup_{t\downarrow 0}|X(s+t)-X(s)|\wedge|X(s)-X(s-t)|/\psi(t)$, one only needs to look at the convergence of $\int_0^1 (1/t)P(X(t)\geqslant c\psi(t))^2\,\mathrm{d}t$, c is a constant. When $\psi(t) = t^{1/(\beta+\varepsilon)}$, note that $g(u) \leqslant u^{\beta+\varepsilon/2}$ for large u, one has that for any constant $c > 0$,

$$\int_0^1 \frac{1}{t}P(X(t) > c\psi(t))^2\,\mathrm{d}t$$

$$\leqslant \int_0^1 \frac{1}{t}\left(tg\left(\frac{1}{c\psi(t)}\right)\right)^2\mathrm{d}t \leqslant K\int_0^1 \frac{1}{t}\cdot t^{2-2(\beta+(\varepsilon/2))/(\beta+\varepsilon)}\,\mathrm{d}t < \infty,$$

where K is a constant. So using Theorems 2 and 3 in Fristedt and Taylor (1992), for fixed $s \in [0,1]$,

$$\limsup_{t\downarrow 0}\frac{|X(s+t)-X(s)|\wedge|X(s)-X(s-t)|}{t^{1/(\beta+\varepsilon)}}=0 \text{ a.s.},$$

which implies that $\bar{d}(\mu, x) \leqslant \beta$ a.s.

Since $\operatorname{Dim} X([0,1])=\beta$ a.s. (Pruitt and Taylor, 1996), one has again by the results in Fristedt and Taylor (1992) that for fixed $s\in[0,1]$,

$$\limsup_{t\downarrow 0}\frac{|X(s+t)-X(s)|\wedge|X(s)-X(s-t)|}{t^{1/(\beta-\varepsilon)}}=\infty \text{ a.s.},$$

which implies that $\bar{d}(\mu,x)\geqslant\beta-\varepsilon$. Finally we have proved, using a Fubini argument,

Theorem 3.2. *For a general subordinator, with probability one,*

$$\bar{d}(\mu,x)=\beta \quad \text{for } \mu\text{-a.e. } x.$$

We will deduce a uniform result for $\underline{d}(\mu,x)$ from the corresponding result involving $h(t)$. These results seem to require condition (2.8) for some $\lambda>0$.

Theorem 3.3. *Let $h(r)=1/g(1/r)$, g satisfies (2.8) for some $\lambda>0$, then with probability one,*

$$\liminf_{r\downarrow 0}\frac{\log\mu(x-r,x+r)}{\log h(r)}=1,\quad \forall x\in X([0,1]).$$

Proof. Let $\bar{f}$ be defined as in (2.2). Note that for any $\varepsilon>0$,

$$[h(r)]^{1/(1-\varepsilon)}<\bar{f}(r)<h(r)\log\frac{1}{r},\quad r\leqslant r_0=r_0(\varepsilon). \tag{3.5}$$

Using Theorem 1 in Fristedt and Pruitt (1972) one has that for a.s. ω,

$$\limsup_{r\downarrow 0}\sup_{t\in[0,1]}\frac{\mu(X(t)-r,X(t)+r)}{\bar{f}(r)}=c \text{ a.s.},\quad c \text{ constant}.$$

So for every t and $r\leqslant r_0$, $\mu(X(t)-r,X(t)+r)\leqslant 2c\bar{f}(r)$. According to (3.5), there exists a δ such that

$$\begin{aligned}\frac{\log\mu(X(t)-r,X(t)+r)}{\log(1/g(1/r))}&\geqslant\frac{\log 2c}{\log(1/g(1/r))}+\frac{\log\bar{f}(r)}{\log(1/g(1/r))}\\&\geqslant 1-\delta+\frac{\log\log 1/r}{\log(1/g(1/r))},\end{aligned}$$

but $g(1/r)<(1/r)^{\beta+\delta}$, when r is sufficiently small. So

$$\liminf_{r\downarrow 0}\frac{\log\mu(X(t)-r,X(t)+r)}{\log(1/g(1/r))}\geqslant 1,\quad \forall t>0.$$

On the other hand, Lemma 2.3 gives that for a.s. ω,

$$\sup_{t\in[0,1]}\limsup_{r\downarrow 0}\frac{\mu(X(t)-r,X(t)+r)}{h(r)}>0.$$

So for a.s. ω,

$$\liminf_{r\downarrow 0}\frac{\log\mu(X(t)-r,X(t)+r)}{\log h(r)}\leqslant 1,\quad \forall x=X(t),\ t\in[0,1],$$

we have completed the proof of this theorem. $\square$

Lemma 2.3 gives us that for a.s. ω, for all $t \in [0,1]$, there exists $c = c(\omega)$ such that

$$\mu(X(t) - r, X(t) + r) \geqslant ch(r) \geqslant r^{\sigma+\varepsilon}, \quad \text{for infinitely many small } r,$$

so $\underline{d}(\mu, x) \leqslant \sigma + \varepsilon$ a.s., for all $x \in X([0,1])$.

By the results in Fristedt and Pruitt (1972), we know there exists a constant $c_1 > 0$ such that,

$$\limsup_{r\downarrow 0} \sup_{x\in X[0,1]} \frac{\mu(x-r, x+r)}{\bar{f}(r)} \leqslant c_1 \text{ a.s.},$$

which means for a.s. ω, there exists $r_0 = r_0(\omega)$ such that when $r \leqslant r_0$, $\mu B(x,r) \leqslant c_1 f(r)$, $\forall x \in X[0,1]$. But we know $\bar{f}(r) \leqslant r^{\sigma-\varepsilon}$ for r small, so

$$\liminf_{r\downarrow 0} \frac{\log \mu(x-r, x+r)}{\log r} \geqslant \sigma - \varepsilon.$$

We have obtained

Theorem 3.4. *When g satisfies* (2.8) *for some $\lambda > 1$, with probability one*

$$\underline{d}(\mu, x) = \sigma, \quad \forall x \in X([0,1]).$$

The uniform result is likely to be true without condition (2.8), but we cannot complete a general argument. However we did not need (2.8) for the lower bound, so we state

Lemma 3.5. *For a general subordinator, with probability one*

$$\underline{d}(\mu, x) \geqslant \sigma \ \forall x \in X([0,1]).$$

4. The spectrum of $\bar{d}(\mu, x)$

Lemma 4.1. *For a.s. $\omega, \bar{d}(\mu, x) \leqslant 2\beta$, $\forall x \in X([0,1])$.*

Proof. It is sufficient to show that for any $\varepsilon > 0$, for a.s. ω,

$$\liminf_{r\downarrow 0} \frac{\mu(x-r, x+r)}{r^{2\beta+\varepsilon}} = \infty, \quad \forall x \in X([0,1]).$$

In fact, if we let

$$\mathscr{G}_n = \{I_{n,k} = [k2^{-n}, (k+1)2^{-n}], \quad k = 1, \ldots, 2^n : I_{n,k} \cap X([0,1]) \neq \emptyset,$$
$$\mu([(k-1)2^{-n}, (k+2)2^{-n}]) < 2^{-n(2\beta+\varepsilon)}\},$$

then by the strong Markov property of X and Lemma 2.1 we have

$$P(I_{n,k} \in \mathscr{G}_n) \leqslant c_1 2^{-n(2\beta+\varepsilon)} \cdot 2^{n(\beta+\varepsilon/2)} \cdot P(X \text{ hits } I_{n,k}),$$

where c_1 is a constant. But by Lemma 2.2 one knows that

$$E(\text{number of the intervals with length } 1/2^n \text{ hit by } X) \leqslant c_2 2^{n(\beta+\varepsilon/2)},$$

c_2 is a constant. So

$$E(\text{number of } \mathscr{G}_n) \leqslant c_3 2^{-(n\varepsilon/2)}, \quad c_3 \text{ is a constant.}$$

By the Borel–Cantelli Lemma, for a.s. ω, there is an integer $n_0 = n_0(\omega)$, such that for any $n \geqslant n_0$, number of $(\mathscr{G}_n)$ is zero, which means

$$\liminf_{r\downarrow 0} \frac{\mu(x-r,x+r)}{r^{2\beta+\varepsilon}} = \infty, \quad \forall x \in X([0,1]). \qquad \square$$

Lemma 4.2. *For a.s.* ω, $\bar{d}(\mu,x) \geqslant \beta$, $\forall x \in X([0,1])$.

Proof. By Lemma 5 in Fristedt and Pruitt (1972) we have when $t \leqslant t_0$,

$$\inf_{s\in[0,1]} \frac{X(s+t)-X(s)}{\bar{h}_\gamma(t)} \geqslant c, \qquad \inf_{s\in[0,1]} \frac{X(s)-X(s-t)}{\bar{h}_\gamma(t)} \geqslant c, \ \ \gamma > 1, \tag{4.1}$$

where c is a constant, $\bar{h}_\gamma(t)$ is defined as in (2.2). (4.1) implies

$$\liminf_{t\downarrow 0} \inf_{s\in[0,1]} \frac{|X(s+t)-X(s)| \wedge |X(s)-X(s-t)|}{\bar{h}_\gamma(t)} \geqslant c. \tag{4.2}$$

But for any fixed $\varepsilon > 0$, there exists a sequence $u_n \to \infty$ such that $g(u_n) > u_n^{\beta-\varepsilon}$. So $\eta(v_n) := g^{-1}(v_n) < v_n^{1/(\beta-\varepsilon)}$, $v_n = u_n^{\beta-\varepsilon}$. Let $\{t_n\}_{n\geqslant 1}$ be the sequence such that $v_n = \gamma t_n^{-1}|\log t_n|$, then

$$\eta(\gamma t_n^{-1}|\log t_n|) < \gamma^{1/(\beta-\varepsilon)} t_n^{-1/(\beta-\varepsilon)} |\log t_n|^{1/(\beta-\varepsilon)},$$

therefore

$$\bar{h}_\gamma(t_n) \geqslant \gamma^{-1/(\beta-\varepsilon)} t_n^{1/(\beta-2\varepsilon)}, \quad n \text{ large.}$$

Applying this inequality in (4.2) we have

$$\liminf_{n\to\infty} \inf_{s\in[0,1]} \frac{|X(s+t_n)-X(s)| \wedge |X(s)-X(s-t_n)|}{t_n^{1/(\beta-2\varepsilon)}} \geqslant c', \quad c' \text{ is a constant.} \tag{4.3}$$

But for any x in $E_{\beta,\varepsilon} := \{x \in X([0,1])\colon \bar{d}(\mu,x) < \beta - 3\varepsilon\}$, one has $\mu(x-r.x+r) > r^{\beta-3\varepsilon}$ for r small, which implies

$$|X(s+t)-X(s)| \wedge |X(s)-X(s-t)| < t^{1/(\beta-3\varepsilon)}, \quad \forall t \text{ small,}$$

which contradicts (4.3). So $E_{\beta,\varepsilon} = \emptyset$ a.s. Letting $\varepsilon \to 0$, we obtain $\bar{d}(\mu.x) \geqslant \beta$, $\forall x \in X([0,1])$. $\square$

Combining Lemmas 4.1 and 4.2 we deduce

Theorem 4.3. *For a.s.* $\omega, \beta \leqslant \bar{d}(\mu,x) \leqslant 2\beta$, $\forall x \in X([0,1])$.

Define

$$A_\delta = \{x \in X([0,1])\colon \bar{d}(\mu,x) \geqslant \delta\}, \quad \beta \leqslant \delta \leqslant 2\beta.$$

Applying the methods in Horowitz (1968), Hu and Taylor (1997) and using Theorem 2.1 in Dembo et al. (1999), we can show that

$$\left(\frac{2\beta-\delta}{\delta}\right)\cdot\sigma\leqslant\dim A_\delta\leqslant\left[\left(\frac{2\beta-\delta}{\delta}\right)\right]\wedge\sigma \text{ a.s.} \tag{4.4}$$

We do not give details for (4.4) since we are not able to find $\dim A_\delta$ precisely. Instead we look at the spectrum in the time set.

5. Thin spectrum in time

Define

$$R(\mu,x,r)=\frac{\log\mu(x-r,x+r)}{\log h(r)},\quad h(r)=\frac{1}{g(1/r)},$$

$$\bar{R}(\mu,x)=\limsup_{r\downarrow 0}\frac{\log\mu(x-r,x+r)}{\log h(r)},\quad \underline{R}(\mu,x)=\liminf_{r\downarrow 0}\frac{\log\mu(x-r,x+r)}{\log h(r)}.$$

In order to overcome the problem of different scaling behaviour at distinct small r, we consider a spectrum which takes account of this. So we will investigate $R(\mu,x,r)$ instead of $T(\mu,x,r):=\log\mu(x-r,x+r)/\log r$. Recalling (3.3), we know that for μ-a.e. x,

$$\lim_{r\downarrow 0}R(x,r)=1 \text{ a.s.},$$

and Theorem 3.3 tells us that when g satisfies (2.8) for some $\lambda>1$, for a.s. ω,

$$\underline{R}(\mu,x)=1,\quad \forall x\in X([0,1]).$$

The following theorem gives us the range of values of $\bar{R}(\mu,x)$.

Theorem 5.1. *For a general subordinator, with probability one,* $1\leqslant\bar{R}(\mu,x)\leqslant 2$ $\forall x\in X([0,1])$.

Proof. Let $F_{\beta,\varepsilon}=\{x\in X([0,1])\colon \bar{R}(\mu,x)<1-3\varepsilon/\beta\}$. Given $x=X(s)\in F_{\beta,\varepsilon}$, one has $\mu(x-r,x+r)>[h(r)]^{1-(3\varepsilon/\beta)}(\equiv H(r))$, for small r. Note that the inverse function of $H(r)$ is $1/g^{-1}(r^{-1/[1-(3\varepsilon/\beta)]})$, one obtains that

$$|X(s+t)-X(s)|\wedge|X(s)-X(s-t)|<\frac{1}{g^{-1}(t^{-1/[1-(3\varepsilon/\beta)]})}$$

$$<t^{1/[1-(3\varepsilon/\beta)]\cdot 1/(\beta+\varepsilon)},\quad \text{for small } t.$$

But $1/(1-3\varepsilon/\beta)\cdot 1/(\beta+\varepsilon)>1/(\beta-2\varepsilon)$, (4.3) tells us $F_{\beta,\varepsilon}$ is empty. Thus for a.s. ω,

$$\bar{R}(\mu,x)\geqslant 1-\frac{3\varepsilon}{\beta},\quad \forall x\in X([0,1]).$$

Since ε is arbitrary,

$$\bar{R}(\mu,x)\geqslant 1,\quad x\in X([0,1]).$$

Let

$$P_t = P\left(|X(s+t)-X(s)| \wedge |X(s)-X(s-t)| > \frac{1}{g^{-1}(t^{-\alpha})}\right), \quad \alpha > 0.$$

By Lemma 2.1

$$P_t \leqslant \left[\frac{tg(u)}{1-\mathrm{e}^{-u\cdot 1/(g^{-1}(t^{-\alpha}))}}\right]^2 \leqslant ct^{2(1-\alpha)}, \quad \text{taking } u = g^{-1}(t^{-\alpha}),$$

c is a constant. When $0<\alpha<\frac{1}{2}$, $\sum_{n\geqslant 1} 2^n P_{2^{-n}} < \infty$, which means for a.s. ω, $\forall s \in [0,1]$,

$$|X(s+t)-X(s)| \wedge |X(s)-X(s-t)| \leqslant \frac{1}{g^{-1}(t^{-\alpha})}, \quad t \leqslant t_0 = t_0(\omega).$$

Thus

$$\limsup_{t\downarrow 0} \frac{|X(s+t)-X(s)| \wedge |X(s)-X(s-t)|}{1/(g^{-1}(t^{-\alpha}))} \leqslant 1, \quad \forall s \in [0,1],$$

which implies

$$\liminf_{r\downarrow 0} \frac{\mu(X(s)-r, X(s)+r)}{[1/g(1/r)]^{1/\alpha}} \geqslant 1, \quad \forall s \in [0,1].$$

Therefore $\bar{R}(\mu,x) \leqslant 1/\alpha\ \forall x \in X([0,1])$. By letting α increase to $\frac{1}{2}$, we have proved that

$$\bar{R}(\mu,x) \leqslant 2\ \forall x \in X([0,1]). \qquad \square$$

We now move to the time set of exceptional points, using the same construction as in Hu and Taylor (1997) to produce a γ-thin subset of the support $X([0,1])$ (Given $\gamma > 1$, a set is said to be γ-thin at x if there is a sequence r_i such that $[B(x,r_i) - B(x,r_i^{\gamma})] \cap A = \emptyset$, if A is γ-thin at every point $x \in A$, then A is said to be a γ-thin set). In Hu and Taylor (1997) we needed quite complicated calculations to obtain the lower bound. However Dembo et al. (1999) have produced a new technique for establishing the lower bound for such random sets of lim sup type. Then Khoshnevisan et al. (1999) have developed a better method for completing the argument using an appropriate version of the Baire category theorem. We use the methods of Khoshnevisan et al. (1999) suitably modified because their conditions are only satisfied on a subsequence.

First we consider the sets

$$T_\gamma = \{t \in [0,1]\colon \bar{R}(\mu, X(t)) \geqslant \gamma\}, \quad 1 \leqslant \gamma \leqslant 2.$$

We will approximate T_γ by sets of lim sup type. Let $\mathscr{F}_n$ be the collection of dyadic intervals $I_{j,n} = [j/2^n, (j+1)/2^n]$ such that

$$X\left(\frac{j}{2^n}\right) - X\left(\frac{j-1}{2^n}\right) \geqslant \frac{1}{g^{-1}(2^{n/\gamma})}, \quad X\left(\frac{j+2}{2^n}\right) - X\left(\frac{j+1}{2^n}\right) \geqslant \frac{1}{g^{-1}(2^{n/\gamma})}.$$

These conditions imply that, for all $s \in I_{j,n}$,

$$X(s) - X\left(\frac{j-1}{2^n}\right) \geqslant \frac{1}{g^{-1}(2^{n/\gamma})}, \quad X\left(\frac{j+2}{2^n}\right) - X(s) \geqslant \frac{1}{g^{-1}(2^{n/\gamma})}.$$

Now putting $A = \limsup_{n\to\infty}(\bigcup_{I_{j,n}\in\mathscr{F}_n} I_{j,n})$, we can easily check that

$$A \subset A_\gamma := \left\{ s \in [0,1]\colon \limsup_{t\downarrow 0} \frac{|X(s+2t)-X(s)| \wedge |X(s)-X(s-2t)|}{1/g^{-1}(t^{-1/\gamma})} \geqslant 1 \right\}$$

and $A_{\gamma+\varepsilon} \subset T_\gamma$ for all $\varepsilon > 0$.

Now we need to estimate the probability that $I_{j,n} \in \mathscr{F}_n$.

Lemma 5.1. *Let $p_n = P(I_{j,n} \in \mathscr{F}_n)$, then there is a constant c_1 such that*

$$p_n \leqslant c_1 \left(\frac{1}{2^n}\right)^{2-(2/\gamma)} \tag{5.1}$$

and given $\varepsilon > 0$, there exists a subsequence $\{n_i\}_{i\geqslant 1}$ depending on ε such that

$$p_{n_i} \geqslant c_2 \left(\frac{1}{2^{n_i}}\right)^{2-2(1-(2\varepsilon/\beta))1/\gamma}, \quad c_2 \text{ a constant.} \tag{5.2}$$

Proof. By Lemma 2.1 we have (5.1) immediately.

On the other hand, note that $P(X(s+t)-X(s) > \lambda) \approx tv(\lambda,\infty)$, whenever this is small. So

$$p_n \geqslant c_3 \left[\frac{1}{2^n} v\left(\frac{1}{g^{-1}(2^{n/\gamma})},\infty\right)\right]^2, \quad c_3 \text{ a constant.} \tag{5.3}$$

By (2.6) we know that there exists a subsequence n_i such that

$$v\left(\frac{1}{g^{-1}(2^{(n_i/\gamma)})},\infty\right) \geqslant [g^{-1}(2^{(n_i/\gamma)})]^{\beta-\varepsilon}, \tag{5.4}$$

but $g(\lambda) < \lambda^{\beta+\varepsilon}$, for λ large enough, so

$$g^{-1}(2^{n_i/\gamma}) > [2^{n_i/\gamma}]^{1/(\beta+\varepsilon)}. \tag{5.5}$$

Combining (5.3)–(5.5) we obtain (5.1). $\square$

Theorem 5.2. *For a.s. ω, $\dim T_\gamma = (2/\gamma) - 1$, $1 \leqslant \gamma \leqslant 2$.*

Proof. We only prove the case when $1 < \gamma < 2$, other cases are trival.

The results stated in Khoshnevisan et al. (1999) seem to require a condition (Condition 4) which is not satisfied unless $\sigma = \beta$. We discussed with Peres how to get round this difficulty. The first step is to deduce from Lemma 5.1 that

$$\limsup_{n\to\infty} \frac{1}{n} \log_2 p_n^{-1} = 2 - \frac{1}{\gamma}. \tag{5.6}$$

This implies that there is a sequence $n_i \to \infty$ for which

$$\lim_{i\to\infty} \frac{1}{n_i} \log_2 p_{n_i}^{-1} = 2 - \frac{2}{\gamma}. \tag{5.7}$$

We note that there is no problem with Condition 5 in Khoshnevisan et al. (1999) since the random variables $Z_n(I), Z_n(J)$ are completely independent whenever $I = I_{j,n}$, $J = I_{k,n}$ and $|j-k| \geqslant 4$. In fact Condition 5_* holds.

Now in Theorem 3.1 of Khoshnevisan et al. (1999) we use a target set E for which the packing dimension $\operatorname{Dim} E$ remains the same, using only intervals of length 2^{-n_i}, for every sequence $n_i \to \infty$. E could be the range of an independent stable subordinator with appropriate index, or E could be a classical Cantor set with the corresponding ratio to replace "middle third". For such sets E, Theorem 3.1 remains valid with (5.6) replacing Condition 4. Furthermore Theorem 3.2 and Corollary 3.3 in Khoshnevisan et al. (1999) are now valid for this kind of very regular E. Taking $E = [0, 1]$, now gives

$$\dim(A) = 1 - \left(2 - \frac{2}{\gamma}\right) = \frac{2}{\gamma} - 1. \tag{5.8}$$

Since $A_{\gamma+\varepsilon} \subset T_\gamma$, we deduce immediately that $\dim T_\gamma \geqslant 2/(\gamma+\varepsilon) - 1$ a.s. Since $\varepsilon > 0$ is arbitrary this implies

$$\dim T_\gamma \geqslant \frac{2}{\gamma} - 1 \text{ a.s..}$$

On the other hand $T_\gamma \subset A_{\gamma-\varepsilon}$ for $\varepsilon > 0$, ans $A_{\gamma-\varepsilon} \subset A$. It follows that

$$\dim T_\gamma \leqslant 2/(\gamma-\varepsilon) - 1 \text{ a.s.} \quad \text{for each } \varepsilon > 0. \qquad \square$$

The arguments in Khoshnevisan et al. (1999), or Dembo et al. (1999), in fact show that the packing dimension of T_γ is full, that is

Theorem 5.3. *For almost all* ω,

$$\operatorname{Dim} T_\gamma = 1 \quad \textit{for } 1 \leqslant \gamma \leqslant 2.$$

Remark. In the stable case, studied in Hu and Taylor (1997), we were not able to find the packing dimension of the exceptional sets. Theorem 5.3 shows that it is full so that, for a subordinator of index α

$$\operatorname{Dim}(\{x\colon \bar{d}(\mu, x) \geqslant \gamma\alpha\}) = \alpha \quad \text{for } 1 \leqslant \gamma \leqslant 2.$$

Now we consider the sets U_γ given by (1.7). In Hu and Taylor (1999) and Dembo et al. (1999) we needed quite complicated potential theory arguments to replace $\geqslant$ by $=$ in the spectrum, because it was necessary to find a gauge function $\phi(s)$ of appropriate dimension $(2/\gamma) - 1$ such that $\phi - m(T_\gamma) > 0$ to be able to deduce that $\phi - m(U_\gamma) > 0$ so that U_γ and T_γ have the same dimension. However, it is pointed out in Khoshnevisan et al. (1999) that this hard argument can be replaced by a "soft" Baire category argument. In Khoshnevisan et al. (1999) the details are given in Theorem 2.3 showing that the inequality of Theorem 2.1 can be replaced by equality. The same method can be applied to Theorem 3.2 and Corollary 3.3 of Khoshnevisan et al. (1999), giving the required result for all sets of lim sup type. This gives our final result.

Theorem 5.4. *For a general subordinator, with probability one, if* $1 \leqslant \gamma \leqslant 2$ *and* T_γ, U_γ *are defined by* (1.6), (1.7), *then*

$$\dim U_\gamma = \dim T_\gamma = \frac{2}{\gamma} - 1, \qquad \operatorname{Dim} U_\gamma = \operatorname{Dim} T_\gamma = 1 \ a.s.$$

and $U_\gamma = \emptyset$ *for* $\gamma < 1$ *or* $\gamma > 2$.

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